

BELTRAMI VECTOR FIELDS WITH POLYHEDRAL SYMMETRIES

GIEDRIUS ALKAUSKAS

ABSTRACT. A vector field B is said to be *Beltrami vector field* (force free-magnetic vector field in physics), if $B \times (\nabla \times B) = 0$. Motivated by our investigations on projective superflows, and as an important side result, we construct two unique 3-dimensional Beltrami vector fields \mathfrak{I} and \mathfrak{J} , such that $\nabla \times \mathfrak{I} = \mathfrak{I}$, $\nabla \times \mathfrak{J} = \mathfrak{J}$, and that both have orientation-preserving icosahedral symmetry (group of order 60). Analogous constructions are done for tetrahedral and octahedral groups of orders 12 and 24, respectively. This construction leads to new notions of *lambent vector field* and *lambent flow*.

1. LAMBENT VECTOR FIELDS AND FLOWS

1.1. The main result. For a 3-dimensional vector field B let, as usual, $\nabla \times B = \text{curl } B$, and let us fix $\phi = \frac{1+\sqrt{5}}{2}$. One of the main results of this paper can be stated immediately.

Theorem 1. *Let us define the vector field $\mathfrak{V} = (\mathfrak{V}_x(x, y, z), \mathfrak{V}_x(y, z, x), \mathfrak{V}_x(z, x, y))$, where*

$$\begin{aligned} \mathfrak{V}_x &= 2x \sin\left(\frac{x}{2}\right) \sin\left(\frac{\phi y}{2}\right) \sin\left(\frac{z}{2\phi}\right) - 2\phi x \sin\left(\frac{x}{2\phi}\right) \sin\left(\frac{y}{2}\right) \sin\left(\frac{\phi z}{2}\right) + 2\phi^{-1} x \sin\left(\frac{\phi x}{2}\right) \sin\left(\frac{y}{2\phi}\right) \sin\left(\frac{z}{2}\right) \\ &+ y \sin z + 2y \cos\left(\frac{x}{2}\right) \cos\left(\frac{\phi y}{2}\right) \sin\left(\frac{z}{2\phi}\right) - 2y \cos\left(\frac{x}{2\phi}\right) \cos\left(\frac{y}{2}\right) \sin\left(\frac{\phi z}{2}\right) \\ &+ z \sin y - 2z \cos\left(\frac{x}{2}\right) \sin\left(\frac{\phi y}{2}\right) \cos\left(\frac{z}{2\phi}\right) + 2z \cos\left(\frac{\phi x}{2}\right) \sin\left(\frac{y}{2\phi}\right) \cos\left(\frac{z}{2}\right), \end{aligned}$$

and the vector field $\mathfrak{W} = (\mathfrak{W}_x(x, y, z), \mathfrak{W}_x(y, z, x), \mathfrak{W}_x(z, x, y))$, where

$$\begin{aligned} \mathfrak{W}_x &= x \cos y - x \cos z \\ &- \sqrt{5} x \cos\left(\frac{x}{2}\right) \cos\left(\frac{\phi y}{2}\right) \cos\left(\frac{z}{2\phi}\right) + \phi x \cos\left(\frac{x}{2\phi}\right) \cos\left(\frac{y}{2}\right) \cos\left(\frac{\phi z}{2}\right) + \phi^{-1} x \cos\left(\frac{\phi x}{2}\right) \cos\left(\frac{y}{2\phi}\right) \cos\left(\frac{z}{2}\right) \\ &- \phi^{-2} y \sin\left(\frac{x}{2}\right) \sin\left(\frac{\phi y}{2}\right) \cos\left(\frac{z}{2\phi}\right) - \phi^2 y \sin\left(\frac{x}{2\phi}\right) \sin\left(\frac{y}{2}\right) \cos\left(\frac{\phi z}{2}\right) + \sqrt{5} y \sin\left(\frac{\phi x}{2}\right) \sin\left(\frac{y}{2\phi}\right) \cos\left(\frac{z}{2}\right) \\ &- \phi^2 z \sin\left(\frac{x}{2}\right) \cos\left(\frac{\phi y}{2}\right) \sin\left(\frac{z}{2\phi}\right) - \phi^{-2} z \sin\left(\frac{\phi x}{2}\right) \cos\left(\frac{y}{2\phi}\right) \sin\left(\frac{z}{2}\right) + \sqrt{5} z \sin\left(\frac{x}{2\phi}\right) \cos\left(\frac{y}{2}\right) \sin\left(\frac{\phi z}{2}\right). \end{aligned}$$

Then

- 1) *The vector field $\mathfrak{I} = (\mathfrak{I}_x, \mathfrak{I}_y, \mathfrak{I}_z) = \mathfrak{V} + \mathfrak{W}$ has an icosahedral symmetry: if we treat \mathfrak{I} as a map $\mathbb{R}^3 \mapsto \mathbb{R}^3$, for any $\zeta \in \mathbb{I}$ (see Subsection 2.2) one has $\zeta^{-1} \circ \mathfrak{I} \circ \zeta = \mathfrak{I}$;*
- 2) *it satisfies $\nabla \times \mathfrak{I} = \mathfrak{I}$;*
- 3) *the Taylor series for \mathfrak{I}_x starts with a degree 6 form $\frac{\varpi}{768}$, where ϖ is given by*

$$\varpi = (5 - \sqrt{5})yz^5 + (5 + \sqrt{5})y^5z - 20y^3z^3 + (10 + 10\sqrt{5})x^2yz^3 + (10 - 10\sqrt{5})x^2y^3z - 10x^4yz.$$

Date: 15 January, 2017.

2010 Mathematics Subject Classification. Primary 37C10, 15Q31.

Key words and phrases. Beltrami vector field, force-free magnetic field, regular polyhedra, Euler's equation, curl, irreducible representations.

The research of the author was supported by the Research Council of Lithuania grant No. MIP-072/2015.

The form ϖ is the numerator of the first coordinate of the icosahedral projective superflow [2], whence the idea comes from.

Note that vector fields \mathfrak{V} and \mathfrak{W} have one additional symmetry. Indeed, let us denote by τ the non-trivial automorphism of the number field $\mathbb{Q}(\sqrt{5})$. Thus, $\tau\phi = -\phi^{-1}$. In the above formulas for \mathfrak{V}_x and \mathfrak{W}_x , let us expand everything in Taylor series, swap y and z , leave x intact, and apply τ summand-wise. We readily obtain

$$\tau \mathfrak{V}_x(x, z, y) = \mathfrak{V}_x(x, y, z), \quad \tau \mathfrak{W}_x(x, z, y) = -\mathfrak{W}_x(x, y, z).$$

1.2. The tetrahedral case: motivation. Let us define

$$\alpha \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \eta \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \delta \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1)$$

These are matrices of order $\alpha^2 = \beta^2 = \eta^2 = \delta^3 = I$. Together they generate the full tetrahedral group $\widehat{\mathbb{T}}$ of order 24, and α, β and δ generate tetrahedral group \mathbb{T} of order 12. In fact, already α and δ generate \mathbb{T} , and α, η and δ generate the whole group $\widehat{\mathbb{T}}$.

A vector field $\mathbf{M}_2 = (2yz, 2xz, 2xy)$ produces the *tetrahedral projective superflow* [1]. This is the unique, up to a scalar multiple, polynomial vector field of degree 2 which has a tetrahedral symmetry \mathbb{T} . Here we encounter the phenomenon that the symmetry *a posteriori* extends to a bigger group $\widehat{\mathbb{T}}$ (see [1], Note 5).

As we will see now, in another direction, this vector field gives rise to another surprising non-rational vector field \mathfrak{T} with few unique properties via the following construction. Namely, the vector field \mathfrak{T} has a tetrahedral symmetry \mathbb{T} , and is equals its own curl.

Since $\nabla \cdot \mathbf{M}_2 = 0$, there exist a vector field $\mathbf{R} = (r_x, r_y, r_z)$ such that $\nabla \times \mathbf{R} = \mathbf{M}_2$. In expanded terms, this reads as

$$yz = \frac{\partial r_z}{\partial y} - \frac{\partial r_y}{\partial z}, \quad xz = \frac{\partial r_x}{\partial z} - \frac{\partial r_z}{\partial x}, \quad xy = \frac{\partial r_y}{\partial x} - \frac{\partial r_x}{\partial y}.$$

The general solution to these equations is given by

$$(r_x, r_y, r_z) = \left(\frac{1}{2}x(z^2 - y^2) + \frac{\partial Q}{\partial x}, \frac{1}{2}y(x^2 - z^2) + \frac{\partial Q}{\partial y}, \frac{1}{2}z(y^2 - x^2) + \frac{\partial Q}{\partial z} \right),$$

where Q is an arbitrary function in three variables x, y, z , with needed smoothness. We want this vector field to satisfy the following three properties (for $n = 3$):

- i) it is n -homogeneous;
- ii) $\nabla \cdot \mathbf{R} = 0$;
- iii) vector field \mathbf{R} has at least a 12-fold symmetry with respect to the group \mathbb{T} . That is, invariance under matrices α, β and δ .

So, Q is 4 homogeneous, and

$$\nabla^2 Q = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) Q = 0.$$

If $Q = 0$, the above vector field has the needed properties. Moreover, its symmetry group is of order 24, and is equal to $\mathbb{T} \times \{I, -I\}$. Let therefore

$$\mathbf{M}_3 = \left(\frac{1}{2}x(z^2 - y^2), \frac{1}{2}y(x^2 - z^2), \frac{1}{2}z(y^2 - x^2) \right).$$

However, this is not the unique choice. We could also choose

$$\begin{aligned} \mathbf{M}_3^{(a)} &= \left(\frac{1}{2}x(z^2 - y^2), \frac{1}{2}y(x^2 - z^2), \frac{1}{2}z(y^2 - x^2) \right) \\ &+ a \left(\frac{x^3}{3} - \frac{x}{2}(y^2 + z^2), \frac{y^3}{3} - \frac{y}{2}(z^2 + x^2), \frac{z^3}{3} - \frac{z}{2}(x^2 + y^2) \right), \quad a \in \mathbb{R}. \end{aligned}$$

Now, as the next step, we will perform the same inverse problem for \mathbf{M}_3 as we did for \mathbf{M}_2 . Namely, we will seek for a vector field \mathbf{M}_4 such that

$$\nabla \times \mathbf{M}_4 = \mathbf{M}_3, \quad \nabla \cdot \mathbf{M}_4 = 0,$$

and $\mathbf{R} = \mathbf{M}_4$ satisfies the above three properties i), ii) and iii) for $n = 4$. And we indeed find

$$\mathbf{M}_4 = \left(-\frac{1}{6}(y^3z + yz^3), -\frac{1}{6}(z^3x + zx^3), -\frac{1}{6}(x^3y + xy^3) \right).$$

This vector field has the same symmetry group as \mathbf{M}_2 . Continuing, we can take

$$\begin{aligned} \mathbf{M}_{2\ell+1} &= \left(\frac{(-1)^\ell}{(2\ell)!}x(y^{2\ell} - z^{2\ell}), \frac{(-1)^\ell}{(2\ell)!}y(z^{2\ell} - x^{2\ell}), \frac{(-1)^\ell}{(2\ell)!}z(x^{2\ell} - y^{2\ell}) \right), \\ \mathbf{M}_{2\ell+2} &= \left(\frac{(-1)^\ell yz}{(2\ell+1)!}(y^{2\ell} + z^{2\ell}), \frac{(-1)^\ell xz}{(2\ell+1)!}(z^{2\ell} + x^{2\ell}), \frac{(-1)^\ell xy}{(2\ell+1)!}(x^{2\ell} + y^{2\ell}) \right). \end{aligned}$$

The above identities holds for $\ell = 0$, with the correct value $\mathbf{M}_1 = \nabla \times \mathbf{M}_2 = 0$. Let

$$\begin{aligned} \mathfrak{V} &= \sum_{\ell=0}^{\infty} \mathbf{M}_{2\ell+2} = \left(z \sin y + y \sin z, x \sin z + z \sin x, y \sin x + x \sin y \right), \\ \mathfrak{W} &= \sum_{\ell=0}^{\infty} \mathbf{M}_{2\ell+1} = \left(x \cos y - x \cos z, y \cos z - y \cos x, z \cos x - z \cos y \right), \\ \mathfrak{Z} &= \mathfrak{V} + \mathfrak{W} = (\mathfrak{V}_x, \mathfrak{V}_y, \mathfrak{V}_z) + (\mathfrak{W}_x, \mathfrak{W}_y, \mathfrak{W}_z) = (\mathfrak{Z}_x, \mathfrak{Z}_y, \mathfrak{Z}_z). \end{aligned}$$

Thus, we have proved the following.

Proposition 1. *Vector fields $\mathfrak{Z}, \mathfrak{V}$ and \mathfrak{W} have these properties:*

- 1s) $\nabla \times \mathfrak{Z} = \mathfrak{Z}$ (this implies $\nabla \cdot \mathfrak{Z} = 0$);
- 2s) \mathfrak{Z} has a group of orientation preserving symmetries of a tetrahedron as a group of its symmetries (invariance under conjugating with matrices α, β and δ);
- 3s) One has

$$\frac{\mathfrak{Z}(xt, yt, zt)}{t^2} \Big|_{t=0} = (2yz, 2xz, 2xy).$$

Further, for vector fields \mathfrak{V} and \mathfrak{W} one has:

$$1vw) \quad \nabla \times \mathfrak{W} = \mathfrak{V}, \quad \nabla \times \mathfrak{V} = \mathfrak{W}, \quad \nabla \cdot \mathfrak{V} = 0, \quad \nabla \cdot \mathfrak{W} = 0.$$

2vw) \mathfrak{V} has the group $\widehat{\mathbb{T}}$ of order 24 as the group of its symmetries; \mathfrak{W} has the group $\mathbb{T} \times \{I, -I\}$ of order 24 as the group of its symmetries (this group is generated by matrices α, β, δ , and $-I$).

The vector field \mathfrak{T} itself is given by (2); see below. The corresponding differential system for a flow reads, for $(x, y, z) = (x(t), y(t), z(t))$, as

$$x' = \mathfrak{T}_x, \quad y' = \mathfrak{T}_y, \quad z' = \mathfrak{T}_z.$$

Let $\mathbf{x} = (x, y, z)$. We call the vector field \mathfrak{T} the *tetrahedral lambent field*, and the corresponding flow $F_{\mathfrak{T}}(\mathbf{x}; t) = (x(t), y(t), z(t))$, $F_{\mathfrak{T}}(\mathbf{x}; 0) = \mathbf{x}$ - the *tetrahedral lambent flow*. Further, we call the vector field \mathfrak{V} *tetrahedral evenly-lambent field*, and \mathfrak{W} - *tetrahedral oddly-lambent field*. The definition of all these will be given in the next Subsection.

If \mathscr{W} is the first integral for the flow with the vector field \mathfrak{W} , then as $\frac{\partial \mathscr{W}}{\partial x} \cdot \mathfrak{W}_x + \frac{\partial \mathscr{W}}{\partial y} \cdot \mathfrak{W}_y + \frac{\partial \mathscr{W}}{\partial z} \cdot \mathfrak{W}_z = 0$. We see that two independent solutions are given by

$$\mathscr{W}^{(1)} = xyz, \text{ and } \mathscr{W}^{(2)} = \text{Ci}(x) + \text{Ci}(y) + \text{Ci}(z), \quad \text{Ci}(x) = - \int_x^\infty \frac{\cos t \, dt}{t}.$$

So, the orbits of this flow are space curves $\{\mathscr{W}^{(1)} = \xi, \mathscr{W}^{(2)} = \chi : \xi, \chi \in \mathbb{R}\}$.

1.3. Generalization: lambent flows. Let ∇^2 be a *vector Laplace operator*, which maps (r_x, r_y, r_z) to $(\nabla^2 r_x, \nabla^2 r_y, \nabla^2 r_z)$. If B is any smooth vector field, then we have a standard identity of vector calculus

$$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B.$$

This, applied to \mathfrak{V} and \mathfrak{W} , gives

$$\mathfrak{V} = -\nabla^2 \mathfrak{V}, \quad \mathfrak{W} = -\nabla^2 \mathfrak{W}.$$

Thus, any coordinate c of \mathfrak{V} or \mathfrak{W} satisfies a (scalar) Helmholtz equation $c = -\nabla^2 c$, which is immediate to check. In the other direction: if $B = -\nabla^2 B$, and $\nabla \cdot B = 0$, then $\nabla \times (\nabla \times B) = B$. Note that the last expression is curl notation-free. Thus, we may proceed with the following definition.

Definition 1. Let $\Gamma \hookrightarrow \text{GL}(3, \mathbb{R})$ be an exact irreducible representation of a finite group, $\mathbf{x} = (x, y, z)$. Suppose, a vector field F satisfies these properties.

- i) $\nabla \times F = F$.
- ii) All three coordinates of F are entire functions, and

$$F(\mathbf{x}) = \sum_{\ell=2}^{\infty} \mathbf{M}_\ell(\mathbf{x}),$$

where $\mathbf{M}_\ell(\mathbf{x})$ is an ℓ -homogeneous polynomial component of $F(\mathbf{x})$.

- iii) if Γ is the group in consideration, and if F is treated as a map $\mathbb{R}^3 \mapsto \mathbb{R}^3$, then $\epsilon^{-1} \circ F \circ \epsilon = F$ for any $\epsilon \in \Gamma$. Of course, this implies that $\epsilon^{-1} \circ \mathbf{M}_\ell \circ \epsilon = \mathbf{M}_\ell$ for any $\epsilon \in \Gamma$ and $\ell \geq 2$, too.
- iv) There exists $\ell \geq 2$ such that the collection $\mathbf{M}_\ell = (p_x, p_y, p_z)$, up to a multiplication by a scalar, is defined by the second half of the property iii) uniquely.

Then the field F is called *lambent field* with symmetry Γ . Corresponding term applies to a flow generated by this field. If we drop the condition iv), the corresponding vector field is called *weakly lambent field* with symmetry Γ .

The fields $G = \frac{1}{2}(F(\mathbf{x}) + F(-\mathbf{x}))$ and $G = \frac{1}{2}(F(\mathbf{x}) - F(-\mathbf{x}))$ we call *evenly-lambent field* with symmetry Γ , and *oddly-lambent field* with symmetry Γ , respectively. These satisfy $\nabla^2 G = -G$, $\nabla \cdot G = 0$.

Thus, as the most relaxed version of these definitions, applicable to all dimensions, we propose the following.

Definition 2. Let $\Gamma \hookrightarrow \text{GL}(n, \mathbb{R})$ be an exact irreducible representation of a finite group. Suppose, a vector field F satisfies these properties:

- i) $\nabla^2 F = -F$, $\nabla \cdot F = 0$;
- ii) All coordinates of F are entire functions;
- iii) if Γ is the group in consideration, and if F is treated as a map $\mathbb{R}^n \mapsto \mathbb{R}^n$, then $\epsilon^{-1} \circ F \circ \epsilon = F$ for any $\epsilon \in \Gamma$.

Then such a vector field is called *sparkling vector field* with symmetry Γ .

Now we are in a position to formulate the main result of this paper.

Theorem 2. Let us define vector fields $\mathfrak{T} = (\mathfrak{T}_x, \mathfrak{T}_y, \mathfrak{T}_z)$ and $\mathfrak{D} = (\mathfrak{D}_x, \mathfrak{D}_y, \mathfrak{D}_z)$ as follows:

$$\begin{cases} \mathfrak{T}_x = z \sin y + y \sin z + x \cos y - x \cos z, \\ \mathfrak{T}_y = x \sin z + z \sin x + y \cos z - y \cos x, \\ \mathfrak{T}_z = y \sin x + x \sin y + z \cos x - z \cos y, \end{cases} \quad (2)$$

and

$$\begin{cases} \mathfrak{D}_x = y \sin z - z \sin y + x \cos y - 2 \sin x + x \cos z, \\ \mathfrak{D}_y = z \sin x - x \sin z + y \cos z - 2 \sin y + y \cos x, \\ \mathfrak{D}_z = x \sin y - y \sin x + z \cos x - 2 \sin z + z \cos y. \end{cases} \quad (3)$$

For the tetrahedral group (of order 12), a family of lambent fields is given by $\mathfrak{T} + a\mathfrak{D}$, $a \in \mathbb{R}$. For the octahedral group (of order 24) the lambent vector field is given by \mathfrak{D} .

For the icosahedral group \mathbb{I} (of order 60), a 1-parameter family of lambent vector fields is given by $\mathfrak{I} + a\mathfrak{D}$, $a \in \mathbb{R}$. Here \mathfrak{I} is given by Theorem 1, and \mathfrak{D} is as follows.

Let us define the vector field $\mathfrak{W}^0 = (\mathfrak{W}_x^0(x, y, z), \mathfrak{W}_x^0(y, z, x), \mathfrak{W}_x^0(z, x, y))$, where its even part is defined by

$$\begin{aligned} \mathfrak{W}_x^0 &= -\phi x \sin\left(\frac{x}{2}\right) \sin\left(\frac{\phi y}{2}\right) \sin\left(\frac{z}{2\phi}\right) + \phi^2 x \sin\left(\frac{x}{2\phi}\right) \sin\left(\frac{y}{2}\right) \sin\left(\frac{\phi z}{2}\right) - x \sin\left(\frac{\phi x}{2}\right) \sin\left(\frac{y}{2\phi}\right) \sin\left(\frac{z}{2}\right) \\ &- \phi^2 y \sin z - \frac{1+3\sqrt{5}}{2} \cos\left(\frac{x}{2}\right) \cos\left(\frac{\phi y}{2}\right) \sin\left(\frac{z}{2\phi}\right) \\ &- \phi^3 y \cos\left(\frac{x}{2\phi}\right) \cos\left(\frac{y}{2}\right) \sin\left(\frac{\phi z}{2}\right) - \frac{5+\sqrt{5}}{2} y \cos\left(\frac{\phi x}{2}\right) \cos\left(\frac{y}{2\phi}\right) \sin\left(\frac{z}{2}\right) \\ &+ z \sin y + z(3+2\sqrt{5}) \cos\left(\frac{x}{2}\right) \sin\left(\frac{\phi y}{2}\right) \cos\left(\frac{z}{2\phi}\right) \\ &+ \phi^{-1} z \cos\left(\frac{\phi x}{2}\right) \sin\left(\frac{y}{2\phi}\right) \cos\left(\frac{z}{2}\right) + \frac{5+\sqrt{5}}{2} z \cos\left(\frac{x}{2\phi}\right) \sin\left(\frac{y}{2}\right) \cos\left(\frac{\phi z}{2}\right), \end{aligned}$$

and its odd part $\mathfrak{W}^0 = (\mathfrak{W}_x^0(x, y, z), \mathfrak{W}_x^0(y, z, x), \mathfrak{W}_x^0(z, x, y)) = \nabla \times \mathfrak{W}^0$. Then the vector field $\mathfrak{Y} = \mathfrak{V}^0 + \mathfrak{W}^0$ has an icosahedral symmetry, and satisfies $\nabla \times \mathfrak{Y} = \mathfrak{Y}$. The Taylor series for \mathfrak{Y}_x starts from the degree 10, and is given by $\frac{\varpi_0}{46448640}$, where ϖ_0 is as in (10).

Concerning this, it is apt to pose the following problem.

Problem. *Classify all lambent and sparkling vector fields. Describe dynamic and qualitative properties of lambent flows.*

In this paper we only present examples of lambent vector fields, based on the solution to the Helmholtz equation which are of the form $L(x, y, z) \cos(l(x, y, z))$, where L and l are linear forms; see Lemma 1. However, for any homogeneous harmonic polynomial $H(y, z)$ in two variables, the function $H(y, z) \cos(x)$ also solves the Helmholtz equation. There are two linearly independent harmonic polynomials of each degree $d \geq 1$, and this is in accordance with Lemma 1, where only the case $d = 1$ is considered. Thus, the variety of lambent vector field is very likely much richer than found here.

1.4. Background. A vector field B is said to be *Beltrami vector field*, if $B \times (\nabla \times B) = 0$. So, for a certain scalar function α , $\nabla \times B = \alpha B$. In this paper we deal with a special important case $\alpha \equiv 1$.

There are many solutions to the equation $\nabla \times B = B$ to be found in the literature. For example, it is known that if the vector field is of constant length 1, then in some Euclidean coordinates one has

$$B = (\sin z, \cos z, 0).$$

Another solution to $\nabla \times B = B$ is the *Lundquist solution* given by, in cylindrical coordinates (r, φ, z) , $(x, y, z) = (r \cos \varphi, r \sin \varphi, z)$, by

$$(0, J_1(r), J_0(r)),$$

where J_* are the Bessel functions.

In [5], Section 7.2, the author considers related problem of finding solutions to the equation $\nabla \times B = |B|B$, where $|B|$ is vectors length. One of rational solutions to this is given by

$$\frac{8(xz - y)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial x} + \frac{8(x + yz)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial y} + \frac{4(1 + z^2 - x^2 - y^2)}{(1 + x^2 + y^2 + z^2)^2} \frac{\partial}{\partial z}.$$

Vector fields

$$\mathbf{b} = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos x)$$

give rise to flows which are known as *Arnold-Beltrami-Childress (ABC) flows*. They satisfy $\nabla \times \mathbf{b} = \mathbf{b}$, and have many profound dynamical properties. Symmetry-related questions of these are discussed in [11].

2. OTHER 3-DIMENSIONAL CASES

2.1. **The octahedral case.** Let us define

$$\alpha \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \gamma \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (4)$$

These matrices generate *the octahedral group* \mathbb{O} of order 24.

Consider the vector field

$$\mathbf{M}_4 = \left(\frac{yz}{6}(y^2 - z^2), \frac{zx}{6}(z^2 - x^2), \frac{xy}{6}(x^2 - y^2) \right).$$

This vector field has a symmetry group \mathbb{O} , and is the unique, up to multiplication by a scalar, vector field of degree 4 with this property (see [1], Subsection 5.1). As we will shortly see, it gives rise to the octahedral lambent vector field \mathfrak{D} .

We have

$$\mathbf{M}_3 = \nabla \times \mathbf{M}_4 = \left(\frac{x^3}{3} - \frac{x}{2}(y^2 + z^2), \frac{y^3}{3} - \frac{y}{2}(z^2 + x^2), \frac{z^3}{3} - \frac{z}{2}(x^2 + y^2) \right). \quad (5)$$

This vector field has the full octahedral symmetry $\widehat{\mathbb{O}}$ of order 48, obtained from \mathbb{O} by adjoining $-I$. The curious fact is that \mathbf{M}_3 does not satisfy the condition iv) of the definition, though \mathbf{M}_4 does. Indeed, all vector fields

$$\left(ax^3 - bx(y^2 + z^2), ay^3 - by(z^2 + x^2), az^3 - bz(x^2 + y^2) \right)$$

have the full octahedral symmetry, and this is a 2-parameter family, rather than 1-parameter, as required by the property iv). However, only for $3a = 2b$ it is solenoidal.

Now, since $\nabla \times \mathbf{M}_3 = \mathbf{0}$, we may act completely analogously as in the tetrahedral case. This, for example, gives

$$\begin{aligned} \mathbf{M}_{2\ell+1} &= \left(\frac{(-1)^\ell x}{(2\ell)!} \left(y^{2\ell} + z^{2\ell} - \frac{2x^{2\ell}}{2\ell+1} \right), \frac{(-1)^\ell y}{(2\ell)!} \left(z^{2\ell} + x^{2\ell} - \frac{2y^{2\ell}}{2\ell+1} \right), \frac{(-1)^\ell z}{(2\ell)!} \left(x^{2\ell} + y^{2\ell} - \frac{2z^{2\ell}}{2\ell+1} \right) \right); \\ \mathbf{M}_{2\ell+2} &= \left(\frac{(-1)^\ell yz}{(2\ell+1)!} (z^{2\ell} - y^{2\ell}), \frac{(-1)^\ell zx}{(2\ell+1)!} (x^{2\ell} - z^{2\ell}), \frac{(-1)^\ell xy}{(2\ell+1)!} (y^{2\ell} - x^{2\ell}) \right). \end{aligned}$$

This is also valid for $\ell = 0$, since $\mathbf{M}_2 = \mathbf{M}_1 = \mathbf{0}$. So, if we define again $\mathfrak{V} = \sum_{\ell=0}^{\infty} \mathbf{M}_{2\ell+2}$,

$\mathfrak{W} = \sum_{\ell=0}^{\infty} \mathbf{M}_{2\ell+1}$, then

$$\begin{aligned} \mathfrak{V} &= \left(y \sin z - z \sin y, z \sin x - x \sin z, x \sin y - y \sin x \right), \\ \mathfrak{W} &= \left(x \cos y - 2 \sin x + x \cos z, y \cos z - 2 \sin y + y \cos x, z \cos x - 2 \sin z + z \cos y \right), \\ \mathfrak{D} &= \mathfrak{V} + \mathfrak{W} = (\mathfrak{V}_x, \mathfrak{V}_y, \mathfrak{V}_z) + (\mathfrak{W}_x, \mathfrak{W}_y, \mathfrak{W}_z) = (\mathfrak{D}_x, \mathfrak{D}_y, \mathfrak{D}_z). \end{aligned}$$

Thus, the octahedral lambent vector field is given by (3), which gives rise to the octahedral lambent flow.

2.2. The icosahedral case. As already defined, let $\phi = \frac{1+\sqrt{5}}{2}$, and define

$$\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \frac{1}{2} & -\frac{\phi}{2} & \frac{1}{2\phi} \\ \frac{\phi}{2} & \frac{1}{2\phi} & -\frac{1}{2} \\ \frac{1}{2\phi} & \frac{1}{2} & \frac{\phi}{2} \end{pmatrix}. \quad (6)$$

These three matrices generate the icosahedral group \mathbb{I} of order 60. Let $\mathbf{M}_6 = (\varpi, \varrho, \sigma)$, where

$$\begin{cases} \varpi = (5 - \sqrt{5})yz^5 + (5 + \sqrt{5})y^5z - 20y^3z^3 + (10 + 10\sqrt{5})x^2yz^3 + (10 - 10\sqrt{5})x^2y^3z - 10x^4yz, \\ \varrho = (5 - \sqrt{5})zx^5 + (5 + \sqrt{5})z^5x - 20z^3x^3 + (10 + 10\sqrt{5})y^2zx^3 + (10 - 10\sqrt{5})y^2z^3x - 10y^4zx, \\ \sigma = (5 - \sqrt{5})xy^5 + (5 + \sqrt{5})x^5y - 20x^3y^3 + (10 + 10\sqrt{5})z^2xy^3 + (10 - 10\sqrt{5})z^2x^3y - 10z^4xy. \end{cases} \quad (7)$$

These are just numerators of a vector field for the icosahedral projective superflow [2]. One has $\zeta^{-1} \circ \mathbf{M}_6 \circ \zeta = \mathbf{M}_6$ for any $\zeta \in \mathbb{I}$, and this is the unique, up to the scalar multiple, 6-homogeneous vector field with this property. Calculations show that $\nabla \times \mathbf{M}_6$ has a full icosahedral symmetry, and $\nabla \times (\nabla \times \mathbf{M}_6) = \mathbf{0}$.

Based on the previous two examples, we will look for the icosahedral lambent vector field $\mathfrak{I} = (\mathfrak{I}_x, \mathfrak{I}_y, \mathfrak{I}_z)$ in an analogous form.

First, we will find evenly-lambent icosahedral vector field. Namely, we will try to find constants a_i , and linear forms L_i, k_i , such that

$$\mathfrak{G} = \mathfrak{V}_x = \sum_i (a_i \cos(k_i) + L_i \sin(k_i)). \quad (8)$$

Two other coordinates \mathfrak{V}_y and \mathfrak{V}_z are obtained from \mathfrak{G} by a cyclic permutation.

First, we know that $\mathfrak{V} = (\mathfrak{V}_x, \mathfrak{V}_y, \mathfrak{V}_z)$ is invariant under conjugation with $\alpha = \text{diag}(-1, -1, 1)$, $\beta\alpha\beta^2 = \text{diag}(1, -1, -1)$, and $\beta^2\alpha\beta = \text{diag}(-1, 1, -1)$. These four matrices, together with the unity I , produce the Klein's fourth group $\mathbb{K} < \mathbb{I}$. This gives

$$-\mathfrak{G}(-x, -y, z) = \mathfrak{G}(x, y, z), \quad \mathfrak{G}(x, -y, -z) = \mathfrak{G}(x, y, z).$$

As the first three linear forms k_i , we just take x, y, z . We do not need $-x, -y$ and $-z$ since \sin and \cos are, respectively, an odd and an even function. Invariance under \mathbb{K} gives, respectively, the following sum as a part of \mathfrak{V}_x in (8):

$$az \sin y + by \sin z.$$

Next, let us define the following 12 linear forms and vectors as follows.

$$\begin{aligned} \ell_{x,0}(x, y, z) &= \frac{1}{2}x + \frac{\phi}{2}y + \frac{1}{2\phi}z, & \mathbf{j}_{x,0} &= \left(\frac{1}{2}, \frac{\phi}{2}, \frac{1}{2\phi} \right) \\ \ell_{x,1}(x, y, z) &= -\frac{1}{2}x + \frac{\phi}{2}y + \frac{1}{2\phi}z, & \mathbf{j}_{x,1} &= \left(-\frac{1}{2}, \frac{\phi}{2}, \frac{1}{2\phi} \right) \\ \ell_{x,2}(x, y, z) &= \frac{1}{2}x - \frac{\phi}{2}y + \frac{1}{2\phi}z, & \mathbf{j}_{x,2} &= \left(\frac{1}{2}, -\frac{\phi}{2}, \frac{1}{2\phi} \right), \\ \ell_{x,3}(x, y, z) &= \frac{1}{2}x + \frac{\phi}{2}y - \frac{1}{2\phi}z, & \mathbf{j}_{x,3} &= \left(\frac{1}{2}, \frac{\phi}{2}, -\frac{1}{2\phi} \right). \end{aligned}$$

Similarly we define 8 other linear functions and vectors by cyclically permuting variables: for example, $\ell_{y,2} = \frac{1}{2}y - \frac{\phi}{2}z + \frac{1}{2\phi}x$, and so on. Orthogonality of the matrix γ tells, for example,

that $\mathbf{j}_{x,2}, \mathbf{j}_{z,1}$ and $\mathbf{i}_{y,0}$ are orthonormal vectors. All these relations amount to the same identities $\phi^2 + \phi^{-2} + 1 = 4$ (unit length), or $\phi - \phi^{-1} - 1 = 0$ (orthogonality). This gives 15 linear forms in total, and this is our complete collection in (8).

Next, if the coordinate of the form (8) is invariant under conjugating with \mathbb{K} , it is of the form

$$\begin{aligned} \mathfrak{G} = & az \sin y + by \sin z \\ & + c \cos \ell_{x,0} - c \cos \ell_{x,3} - c \cos \ell_{x,2} + c \cos \ell_{x,1} \\ & + K(x, y, z) \sin \ell_{x,0} + K(-x, -y, z) \sin \ell_{x,3} + K(-x, y, -z) \sin \ell_{x,2} - K(x, -y, -z) \sin \ell_{x,1} \\ & + d \cos \ell_{y,0} - d \cos \ell_{y,2} - d \cos \ell_{y,1} + d \cos \ell_{y,3} \\ & + L(x, y, z) \sin \ell_{y,0} + L(-x, -y, z) \sin \ell_{y,2} + L(-x, y, -z) \sin \ell_{y,1} - L(x, -y, -z) \sin \ell_{y,3} \\ & + e \cos \ell_{z,0} - e \cos \ell_{z,1} - e \cos \ell_{z,3} + e \cos \ell_{z,2} \\ & + M(x, y, z) \sin \ell_{z,0} + M(-x, -y, z) \sin \ell_{z,1} + M(-x, y, -z) \sin \ell_{z,3} - M(x, -y, -z) \sin \ell_{z,2}. \end{aligned} \tag{9}$$

Here a, b, c, d, e are arbitrary constants, and K, L, M are arbitrary linear forms.

Now, recall that for an evenly lambent vector field, $\nabla^2 \mathfrak{G} = -\mathfrak{G}$. Next, the following Lemma is immediate.

Lemma 1. *Let \mathbf{a} and \mathbf{b} be two 3-dimensional vectors-rows, and $\mathbf{x} = (x, y, z)^T$. The function $\mathbf{ax} \sin(\mathbf{bx})$ is a solution to the Helmholtz equation if and only if $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, and $|\mathbf{b}| = 1$. The same holds for the cos function.*

By a direct inspection, two vectors orthogonal to $\mathbf{j}_{x,0}$ are given by $\mathbf{j}_{z,2}$ and $\mathbf{j}_{y,1}$. So,

$$\begin{aligned} K(x, y, z) &= f \ell_{z,2} + g \ell_{y,1}, \\ L(x, y, z) &= h \ell_{x,2} + i \ell_{z,1}, \\ M(x, y, z) &= j \ell_{y,2} + k \ell_{x,1}. \end{aligned}$$

We therefore have 11 free coefficients at our disposition.

Such a function $\mathfrak{G}(x, y, z)$ is invariant under conjugation with \mathbb{K} and satisfies the Helmholtz equation. We will reduce the amount of free coefficients by requiring that a vector field has an icosahedral symmetry.

2.3. Computation. Now, we will proceed with determining 11 free coefficients.

i) First, we will require that the Taylor coefficient of \mathfrak{G} of degree 6 is exactly equal to ϖ , and Taylor coefficients of degrees ≤ 5 are absent. This gives, with computer calculations performed on MAPLE, 6 linear relations among 11 constants. The free constants turn out to be a, b, c, d and f , while the remaining 6 constants e, g, h, i, j, k are linear functions in a, b, c, d, f . This way we obtain a 5-parameter function \mathfrak{G} , such that its Taylor series starts with ϖ , it is invariant under conjugation with \mathbb{K} , and it satisfies the Helmholtz equation.

ii) As a next step, we require that $\nabla \cdot (\mathfrak{G}(x, y, z), \mathfrak{G}(y, z, x), \mathfrak{G}(z, x, y)) = 0$. This gives two more relations, leaving a, b , and d as free coefficients.

iii) Finally, a vector field $\mathfrak{H} = (\mathfrak{G}(x, y, z), \mathfrak{G}(y, z, x), \mathfrak{G}(z, x, y))$ has the tetrahedral symmetry: it is invariant under conjugation with α and β , as in (6). If it is invariant under conjugation with γ , it has the icosahedral symmetry, and the problem is solved. We have:

$$\begin{aligned}\mathfrak{H} &= \gamma^{-1} \circ (\mathfrak{G}(x, y, z), \mathfrak{G}(y, z, x), \mathfrak{G}(z, x, y)) \circ \gamma(x, y, z) \\ &= \gamma^{-1} \circ (\mathfrak{G}(\ell_{x,2}, \ell_{z,1}, \ell_{y,0}), \mathfrak{G}(\ell_{z,1}, \ell_{y,0}, \ell_{x,2}), \mathfrak{G}(\ell_{y,0}, \ell_{x,2}, \ell_{z,1})) \\ &:= \gamma^{-1}(A, B, C).\end{aligned}$$

Therefore, we finally require that

$$\frac{A}{2} + \frac{\phi B}{2} + \frac{C}{2\phi} = \mathfrak{G}(x, y, z).$$

This gives two more relations, and we thus obtained a 1-parameter family (discarding the homothety, which in this case gives another parameter as a scaling factor, which we discard by the property iv) of the definition) of icosahedral lambent vector fields, with a free parameter being a . This comes as a surprise! For any choice of a we obtain a solution.

We will get a particularly symmetric example due to the following observation. Let η be given by (1). Note that if we conjugate the vector field $(\varpi, \varrho, \sigma)$ with respect to η , and then apply the non-trivial automorphism τ of the number field $\mathbb{Q}(\sqrt{5})$ (which exchanges $\sqrt{5} \leftrightarrow -\sqrt{5}$), we arrive at the same vector field $(\varpi, \varrho, \sigma)$. Now, suppose all the constants a through k in $(\mathfrak{G}(x, y, z), \mathfrak{G}(y, z, x), \mathfrak{G}(z, x, y))$, where the first coordinate is given by (9), are elements from $\mathbb{Q}(\sqrt{5})$, and an invariance under the just described transformation holds. This gives the equation $a = \tau b$. However, the last of the 10 linear equations we obtained reads as

$$b = 1920 - \frac{3}{2}a - \frac{\sqrt{5}}{2}a + 384\sqrt{5}.$$

Let us choose $a = 384 \cdot 2$. This gives $b = 384 \cdot 2$, and thus both requirements are reconciled. This yields $c = d = e = 0$, $f = 384\phi^{-1}$, $g = -384\phi$, $h = 384\phi$, $i = 384 \cdot 1$, $j = -384 \cdot 1$, $k = 384\phi^{-1}$. Thus, if we start from a collection of the coefficients $(2, 2, 0, 0, 0, \phi^{-1}, -\phi, \phi, 1, -1, \phi^{-1})$, we arrive at the vector field $\frac{1}{384}(\varpi, \varrho, \sigma)$, where $(\varpi, \varrho, \sigma)$ is given by (7). Now, (9) gives

$$\begin{aligned}\mathfrak{G} &= 2z \sin y + 2y \sin z \\ &+ (-x + y - z) \sin \ell_{x,0} + (x - y - z) \sin \ell_{x,3} + (x + y + z) \sin \ell_{x,2} - (-x - y + z) \sin \ell_{x,1} \\ &+ (\phi x - y) \sin \ell_{y,0} + (-\phi x + y) \sin \ell_{y,2} + (-\phi x - y) \sin \ell_{y,1} - (\phi x + y) \sin \ell_{y,3} \\ &+ (-\phi^{-1}x + z) \sin \ell_{z,0} + (\phi^{-1}x + z) \sin \ell_{z,1} + (\phi^{-1}x - z) \sin \ell_{z,3} - (-\phi^{-1}x - z) \sin \ell_{z,2}.\end{aligned}$$

Finally, let us collect all functions in each row as factors of x, y, z , and use trigonometric addition formulas. For example,

$$-\sin \ell_{x,0} + \sin \ell_{x,3} + \sin \ell_{x,2} + \sin \ell_{x,1} = 4x \sin\left(\frac{x}{2}\right) \sin\left(\frac{\phi y}{2}\right) \sin\left(\frac{z}{2\phi}\right).$$

Thus we get (in fact, MAPLE does it for us) the first displayed formula in Theorem 1, where $(\mathfrak{V}_x, \mathfrak{V}_y, \mathfrak{V}_z) = \frac{1}{2}(\mathfrak{G}(x, y, z), \mathfrak{G}(y, z, x), \mathfrak{G}(z, x, y))$. For the first coordinate of oddly-lambent vector field, we have $\mathfrak{W}_x = \frac{\partial \mathfrak{V}_z}{\partial y} - \frac{\partial \mathfrak{V}_y}{\partial z}$, and this gives the second displayed formula in Theorem 1.

As we now know, any icosahedral lambent vector field of the form (9) is given by $\mathfrak{I} + a'\mathfrak{J}$, $a' \in \mathbb{R}$. Calculations show that \mathfrak{J} is obtained by specializing

$$(a, b, c, d, e, f, g, h, i, j, k) = \left(1, -\phi^2, 0, 0, 0, \frac{\phi^{-1}}{2}, \frac{\phi^3}{2}, \frac{\phi}{2}, -\frac{\phi^2}{2}, -\frac{1}{2}, -\frac{\phi}{2}\right).$$

Taylor series of \mathfrak{J}_x starts at the degree 10, and is given by $\frac{\varpi_0}{46448640}$, where

$$\begin{aligned} \varpi_0 = & (9 + 9\sqrt{5})x^8yz - (252 + 84\sqrt{5})x^6y^3z + 168x^6yz^3 + (378 + 126\sqrt{5})x^4y^5z - 252x^4yz^5 \\ & - (288 + 72\sqrt{5})x^2y^7z + (1260 + 252\sqrt{5})x^2y^5z^3 - (1260 + 252\sqrt{5})x^2y^3z^5 + (252 + 36\sqrt{5})x^2yz^7 \\ & + (8 - 2\sqrt{5})y^9z + 48\sqrt{5}y^7z^3 - (126 + 126\sqrt{5})y^5z^5 + (120 + 72\sqrt{5})z^7y^3 - (17 + 7\sqrt{5})z^9y. \end{aligned} \quad (10)$$

This gives, with the help of MAPLE, the formula in Theorem 2.

REFERENCES

- [1] G. ALKAUSKAS, Projective superflows. I. <http://arxiv.org/abs/1601.06570>.
- [2] G. ALKAUSKAS, Projective superflows. II. $O(3)$ and the icosahedral group. <http://arxiv.org/abs/1606.05772>.
- [3] G. ALKAUSKAS, Projective superflows. III. Finite subgroups of $U(2)$. <http://arxiv.org/abs/1608.02522>.
- [4] G. ALKAUSKAS, Projective superflows. IV. Arithmetic of the orbits (*in preparation*, 2017).
- [5] D. E. BLAIR, *Riemannian geometry of contact and symplectic manifolds*. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, (2002).
- [6] S. CHANDRASEKHAR, P.C. KENDALL, On force-free magnetic fields. *Astrophys. J.* **126** (1957), 457–460.
- [7] S. CHANDRASEKHAR, On force-free magnetic fields. *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 1–5.
- [8] S. CHANDRASEKHAR, L. WOLTJER, On force-free magnetic fields. *Proc. Nat. Acad. Sci. U.S.A.* **44**, 1958, 285–289.
- [9] J. ETNYRE, R. GHRIST, ROBERT, Contact topology and hydrodynamics. I. Beltrami fields and the Seifert conjecture. *Nonlinearity* **13**(2) (2000) 441–458.
- [10] Q. Z. FENG, On force-free magnetic fields and Beltrami flows. *Appl. Math. Mech. (English Ed.)* **18**(10) (1997), 997–1003.
- [11] P. FRE, A.S. SORIN, Classification of Arnold-Beltrami Flows and their Hidden Symmetries. <https://arxiv.org/abs/1501.04604>.
- [12] R. HIPTMAIR, P.R. KOTIUGA, S. TORDEUX, Self-adjoint curl operators. *Ann. Mat. Pura Appl. (4)*, **191**(3) (2012), 431–457.
- [13] R. KAISER, M. NEUDERT, W. VON WAHL, On the existence of force-free magnetic fields with small nonconstant α in exterior domains. *Comm. Math. Phys.* **211**(1) (2000), 111–136.
- [14] V.V. KRAVCHENKO, On force free magnetic fields. Quaternionic approach. *Math. Methods Appl. Sci.* **28**(4) (2005), 79–386.
- [15] M.A. MACLEOD, The spherical curl transform of a linear force-free magnetic field. *J. Math. Phys.* **39**(3) (1998), 1642–1658.
- [16] E.C. MORSE, Eigenfunctions of the curl in annular cylindrical and rectangular geometry. *J. Math. Phys.* **48**(8) (2007).
- [17] E. R. PRIEST, *Solar Magnetohydrodynamics*. Geophysics and Astrophysics Monographs, Volume 21, Springer (1982).
- [18] K. SAYGILI, Trkalian fields and Radon transformation. *J. Math. Phys.* **51**(3) (2010).
- [19] K. SAYGILI, Trkalian fields: ray transforms and mini-twistors. *J. Math. Phys.* **54**(10) (2013).
- [20] E. TASSI, F. PEGORARO, G. CICOGNA, Solutions and symmetries of force-free magnetic fields. *Physics of Plasmas* **15**(9) (2008).
- [21] S.M. MAHAJAN, Z. YOSHIDA, Double Curl Beltrami Flow: Diamagnetic Structures. *Phys. Rev. Lett.* **81**(22), 4863, November 1998.
- [22] S.M. MAHAJAN, Z. YOSHIDA, Simultaneous Beltrami conditions in coupled vortex dynamics. *J. Math. Phys.* **40**(10), (1999), 5080–5091.

-
- [23] N.A. SALINGAROS, On solutions of the equation $\nabla \times a = ka$. *Journal of Physics A: Mathematical and General*, **19**(3), 1986.
- [24] T. WIEGELMANN, T. SAKURAI, Solar Force-free Magnetic Fields. *Living Reviews in Solar Physics*, **9**(5), (2012).
- [25] L. WOLTJER, A theorem on force-free magnetic fields. *Proc. Nat. Acad. Sci. U.S.A.* **44**, 1958, 489-491.

VILNIUS UNIVERSITY, DEPARTMENT OF MATHEMATICS AND INFORMATICS, NAUGARDUKO 24, LT-03225
VILNIUS, LITHUANIA

E-mail address: giedrius.alkauskas@mif.vu.lt